Unit 5

3D Object Representation

3D object representation is divided into two categories:

a) **Boundary Representations (B-reps):** The boundary representations describe a three-dimensional object as a set of surface that separates the object interior from the environment. E.g. Polyhedron, ellipsoid, aircraft, medical images etc.

- B-reps for single polyhedron satisfy Euler's formula: V-E+F=2

b) **Space Partitioning Representation:** Space partitioning describes interior properties by partitioning the spatial region containing an object into a small, non- overlapping, contiguous solids. E.g. 3D object as Octree representation.

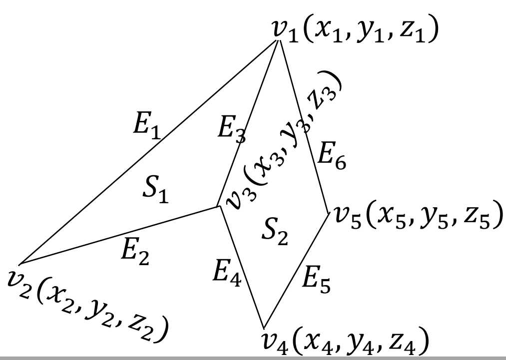
Boundary Representation

1. Polygon Surface

It is most common representation for 3D graphics object. In this representation, a 3D object is represented by a set of surfaces that enclose the object interior. This method simplifies and speeds up the surface rendering and display of the object.
The polygon surfaces are common in design and solid-modeling applications, since wire frame display can be done quickly to give general indication of surface structure. Then realistic scenes are produced by interpolating shading patterns across polygon surface to illuminate.
Polygon surface can be represented by:

a) Polygon Table

In this method, a polygon surface is specified with a set of vertex co-ordinates and associated attributes. Polygon data tables can be organized into two groups: geometrical and attribute tables.
Geometric tables: It contain vertex coordinates and parameters to identify the spatial orientation of polygon surfaces
Attribute table: It gives attribute information for an object (Degree of transparency, surface reflectivity etc.).
Geometric data consists of three tables:
(i) Vertex table: It stores co-ordinate values for each vertex of the object.
(ii) Edge table: It stores the edge information of polygon.
(iii) Surface table: It stores the number of surfaces present in the polygon.



Vertex table

$v_1: x_1, y_1, z_1$
$v_2: x_2, y_2, z_2$
$v_3: x_3, y_3, z_3$
$v_4: x_4, y_4, z_4$
$v_5: x_5, y_5, z_5$

Edge table

$E_1: v_1, v_2$
$E_2: v_2, v_3$
$E_3: v_3, v_1$
$E_4: v_3, v_4$
$E_5: v_4, v_5$
$E_6: v_5, v_1$

Surface table

$S_1: E_1, E_2, E_3$
$S_2: E_3, E_4, E_5, E_6$

The object can be displayed efficiently by using data from tables and processing them for surface rendering and visible surface determination.

b) Polygon table using forward pointer in edge table

For above polygon,

Vertex table

$v_1: x_1, y_1, z_1$
$v_2: x_2, y_2, z_2$
$v_3: x_3, y_3, z_3$
$v_4: x_4, y_4, z_4$
$v_5: x_5, y_5, z_5$

Edge table

$E_1: v_1, v_2, S_1$
$E_2: v_2, v_3, S_1$
$E_3: v_3, v_1, S_1, S_2$
$E_4: v_3, v_4, S_2$
$E_5: v_4, v_5, S_2$
$E_6: v_5, v_1, S_2$

Surface table

$S_1: E_1, E_2, E_3$
$S_2: E_3, E_4, E_5, E_6$

c) Polygon Meshes

A polygon mesh is collection of edges, vertices and polygons connected such that each edge is shared by at most two polygons. An edge connects two vertices and a polygon is a closed sequence of edges. An edge can be shared by two polygons and a vertex is shared by at least two edges.

This method can be used to represent a broad class of solids/surfaces in graphics. A polygon mesh can be rendered using hidden surface removal algorithms. The polygon mesh can be represented by three ways-

- Explicit representation
- Pointers to a vertex list
- Pointers to an edge list

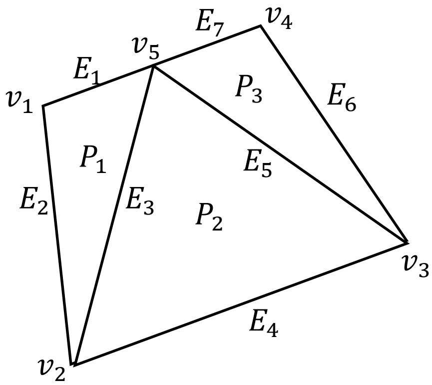
In **Explicit representation**, each polygon is represented by a list of vertex co-ordinates.

$$P = ((x_1, y_1, z_1), (x_2, y_2, z_2), \dots \dots .., (x_n, y_n, z_n))$$

In **Pointers to a vertex list**, each vertex is stored just once, in vertex list

$$V = (v_1, v_2, \dots \dots \dots ., v_n)$$

E.g. A polygon made up of vertices 3, 5, 7, 10 in vertex list be represented as $P_1 = \{3,5,7,10\}$



Representing polygon mesh with each polygon as vertex list.

$P_1 = \{v_1, v_2, v_5\}$

$P_2 = \{v_2, v_3, v_5\}$ $P_3 = \{v_3, v_4, v_5\}$

In **<u>Pointers to an edge list</u>**, we have vertex list V, represent the polygon as a list of pointers to an edge list. Each edge in edge list points to the two vertices in the vertex list. Also to one or two polygon, the edge belongs. Hence, we describe polygon as $P = (E_1, E_2, \dots \dots \dots.., E_n)$ and an edge as $E = (v_1, v_2, P_1, P_2)$ Here if edge belongs to only one polygon, either then P_1 or P_2 is null.

For the mesh given above,

$V = \{v_1, v_2, v_3, v_4, v_5\} = \{((x_1, y_1, z_1), \dots \dots \dots.., (x_5, y_5, z_5)\}$

$E_1 = (v_1, v_5, P_1, N)$ $E_6 = (v_3, v_4, P_3, N)$

$E_2 = (v_1, v_2, P_1, N)$ $E_7 = (v_4, v_5, P_3, N)$

$E_3 = (v_2, v_5, P_1, P_2)$ $P_1 = (E_1, E_2, E_3)$

$E_4 = (v_2, v_3, P_2, N)$ $P_2 = (E_3, E_4, E_5)$

$E_5 = (v_3, v_5, P_1, P_3)$ $P_3 = (E_5, E_6, E_7)$

Here, N represents Null.

d) <u>Plane equation</u>

It this method polygon surface is represented by the equation of plane in the coordinate system. The 3D object is represented through the set of equations. The equation for plane surface can be expressed as

$$Ax + By + Cz + D = 0$$

Where x, y, z is any point on the plane and A, B, C & D are coefficient of plane equation and represents the spatial orientation of the polygon surface in space coordinate system. Hence, the value of coefficient must be known to represent the 3D object.

The value of A, B, C & D can be obtained by solving a set of three plane equation using coordinate of three non-collinear point on plane. Let us assume that three vertices of plane are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). Then,

$Ax_1 + By_1 + Cz_1 + D = 0$

$Ax_2 + By_2 + Cz_2 + D = 0$

$Ax_3 + By_3 + Cz_3 + D = 0$

By Cramer's rule

$$A = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix} \quad B = \begin{vmatrix} x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \end{vmatrix} \quad C = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad D = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

For any points (x, y, z)
If Ax + By + Cz + D ≠ 0, then (x, y, z) is not on the plane.
If Ax + By + Cz + D < 0, then (x, y, z) is inside the plane i. e. invisible side
If Ax + By + Cz + D > 0, then (x, y, z) is lies outside the surface.

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2. Quadratic Surface

Quadric Surface is one of the frequently used 3D objects surface representation. The quadric surface can be represented by a second degree polynomial. This includes:

a. Sphere:

For a set of surface point (x, y, z) spherical surface is represented by equation,

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2$$

Where, (x_c, y_c, z_c) is the center of sphere and r is the radius of the sphere.

b. Ellipsoid:

If (x, y, z) be the any point on ellipsoid with the radius a, b & c along x, y & z axis then represented as;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

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3. Wireframe representation

- If an object may be represented through the collection of points and line then it's called wireframe object.
- It's used to define skeleton of 3D objects in terms of points and lines.
- Mostly use in field of engineering drawings to represent the drawing of structure and missing parts.
- Wireframe represented object requires less memory for storage and fast for display and surface rendering.
- In this method the scenes represented are not realistic.

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4. Blobby objects

Object that don't maintain a fixed shape but changes their surface characteristics during motion or closer to another object are called blobby object. E.g. molecular structure, water droplet, muscle shape in human body etc.

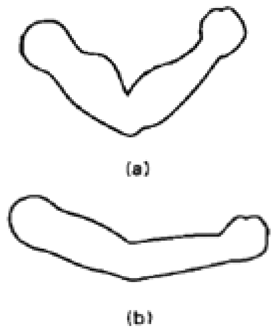

Fig: Blobby muscle in human

Most common function used for blobby object is Gaussian density function. A surface function is defined as:

$$f(x, y, z) = \sum_{k} b_k e^{-a_k r_k^2} - T = 0$$

Where, $r_k^2 = \sqrt{x_k^2 + y_k^2 + z_k^2}$

T= some specified threshold

a, b= parameters used to adjust the amount of blobbiness for individual object.

Advantages

- Can represent organic, blobby or liquid line structures.
- Suitable for modeling natural phenomena like water, human body.
- Surface properties can be easily derived from mathematical equations.

Disadvantages

- Requires expensive computation
- Requires special rendering engine
- Not supported by most graphics hardware

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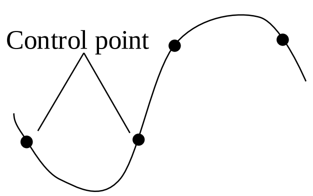
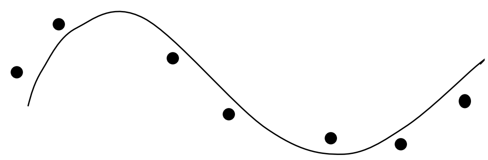
5. Spline representation

- A Spline is a flexible strips used to produce smooth curve through a designated set of points. A curve drawn with these set of points is spline curve. Spline curves are used to model 3D object surface shape smoothly.
- A spline curve is a mathematical representation that allow the user to design and control shape of complex curve and surface.
- Here, user enters a sequence of points called control points & a curve is constructed whose shape closely follows these control points.
- Two types of spline curve:
  - **a. *Interpolating curve:*** A curve that actually passes through each control point.
    - Interpolation curves are commonly used to digitize drawing or to specify animation paths.
  - **b. *Approximating curve:*** A curve that passes near to control point but not necessary through them.
    - Approximation curves are uses as design tools to structure object surfaces.



Interpolating curve


Approximating curve

- Spline curve also used for design of automobile bodies, spacecraft, specification of animation path, home appliance etc.
- The three degree polynomial known as cubic polynomial is typically used for constructing smooth curve in computer graphics because of following reasons:
    a. It is lowest degree polynomial that can support an inflection (a point at which curve crosses its tangent i.e. curve changes from concave to convex).
    b. The curves are smooth like this  and not jumpy like this  .

Spline Specifications:

There are three equivalent methods for specifying a particular spline representation:

a) ***Boundary condition:*** We can state the set of boundary conditions that are imposed on the spline.

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x \qquad 0 \le u \le 1$$

Boundary condition for this curve can be set for x(0), x(1), x'(0) & x'(1). These four conditions are sufficient to determine the values of four coefficient a_x, b_x, c_x & d_x.

b) ***Characterizing matrix:*** We can state the matrix that characterizes the spline.
From the boundary condition, the characterizing matrix for spline is:

$$x(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_x \\ b_x \\ c_x \\ d_x \end{bmatrix}$$

$$= U.C$$

c) ***Blending Function:*** We can state the set of blending functions (or basis functions) that determine how specified geometric constraints on the curve are combined to calculate positions along the curve path.

$$x(u) = \sum_{k=0}^{3} g_k \cdot BF_k(u)$$

g_k = Geometric constrain parameter
$BF_k(u)$=Polynomial blending function

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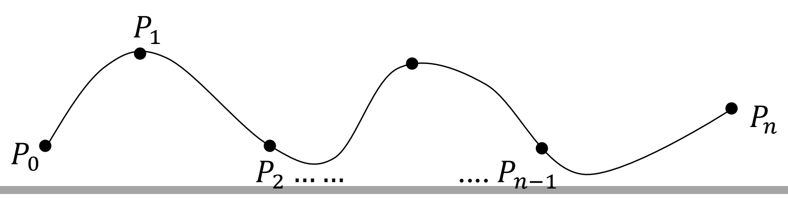
6. <u>Cubic spline</u>

- It is used to set up path for object motions or to provide a representation for an existing object or drawings.
- Compared to higher-order polynomials, cubic splines requires less calculation and memory and they are more stable. Compared to lower-order polynomials, cubic splines are more flexible for modeling arbitrary curve shapes.
- Cubic interpolation spline is obtained by fitting the input points with a piecewise cubic polynomial curve that passes through every control points.

Suppose we have n+1 control points having co-ordinates

$$P_k = (x_k, y_k, z_k) \qquad K = 0, 1, 2, 3, \ldots\ldots\ldots, n$$



A parametric cubic polynomial that is to be fitted between each pair of control points have following equations:

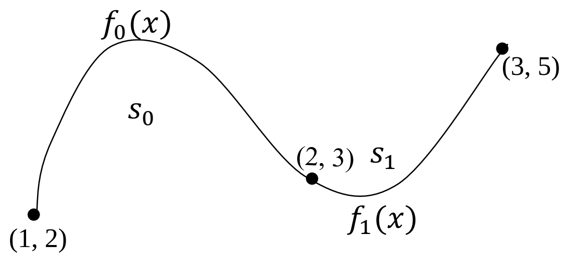
$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$$
$$y(u) = a_y u^3 + b_y u^2 + c_y u + d_y \qquad (0 \le u \le 1)$$
$$z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$$

We need to determine the values of the four coefficients a, b, c, and d in the polynomial representation for each of the *n* curve section. We do this by setting enough boundary conditions at the "joints" between curve sections we can obtain numerical values for all the coefficients.

- Cubic splines are more flexible for modeling arbitrary curve shapes.

Q. Construct a natural cubic spline that passes through (1, 2), (2, 3) & (3, 5) having two segments $f_0(x)$ & $f_1(x)$.

Solution:



<u>For segment s_0</u>

$$f_0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3$$

a) Since it passes through (1, 2)

$a_0 + b_0 + c_0 + d_0 = 2$ ………. (i)

b) Since it passes through (2, 3)

$a_0 + 2b_0 + 4c_0 + 8d_0 = 3$ ………... (ii)

c) Slope of $f_0(x)$ & $f_1(x)$ must be same at (2, 3)

$$\frac{d\,f_0(x)}{dx} = \frac{df_1(x)}{dx}$$

or, $b_0 + 2c_0 x + 3d_0 x^2 = b_1 + 2c_1 x + 3d_1 x^2$

or, $b_0 + 4c_0 + 12d_0 - b_1 - 4c_1 - 12d_1 = 0$ ………….. (iii)

d) Curvature must be same for $f_0(x)$ & $f_1(x)$ at (2, 3)

$$\frac{d\,f_0'(x)}{dx} = \frac{df_1'(x)}{dx}$$

or, $2c_0 + 6d_0 x = 2c_1 + 6d_1 x$

or, $2c_0 - 2c_1 + 6d_0 x - 6d_1 x = 0$

or, $2c_0 - 2c_1 + 12d_0 - 12d_1 = 0$ …………. (iv)

e) $f''(x) = 0$ at (1, 2)

or, $2c_0 + 6d_0 x = 0$

or, $2c_0 + 6d_0 = 0$

or, $c_0 + 3d_0 = 0$……….. (v)

<u>For segment s_1</u>

$$f_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$$

a) Since it passes through (2, 3)

$$a_1 + 2b_1 + 4c_1 + 8d_1 = 3 \text{ .... (vi)}$$

b) Since it passes through (3, 5)

$$a_1 + 3b_1 + 9c_1 + 27d_1 = 5 \text{ .... (vii)}$$

c) $f''(x) = 0$ at (3, 5)

or, $2c_1 + 6d_1x = 0$

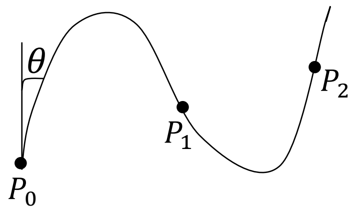
or, $2c_1 + 18d_1 = 0$

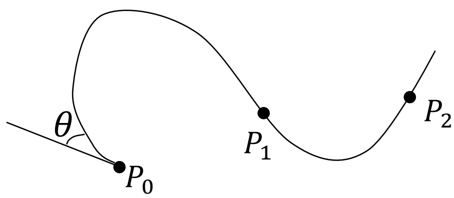
or, , $c_1 + 9d_1 = 0$ ……….. (viii)

Solve these equation and plot the graph.

Hermite Interpolation (Hermite curve)

It is an interpolating piecewise cubic polynomial with a specified tangent at each control point.





If we change the control point at P_0, then the curve will also change, so that angle θ between P_0 and tangent at P_0 will remain constant.

- It has local control over the curve i.e. each curve section depend on its end point only.
- The vector equivalence of Hermite curve is
  $P(u) = au^3 + bu^2 + cu + d$ ………………… (i)

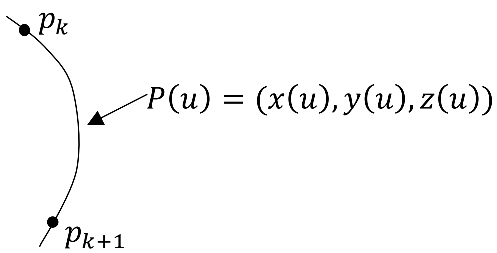
Where, x component of $P(u)$ is

$$x(u) = a_xu^3 + b_xu^2 + c_xu + d_x$$

Similarly, y and z component

$$y(u) = a_yu^3 + b_yu^2 + c_yu + d_y$$
$$z(u) = a_zu^3 + b_zu^2 + c_zu + d_z$$

Let $P(u)$ denotes the parametric cubic point function for the curve section between control point p_k & p_{k+1}.



At p_k, u=0

$\therefore P(0) = p_k$

At p_{k+1}, u=1

$\therefore P(1) = p_{k+1}$

Also let Dp_k & Dp_{k+1} denote the slope at p_k & p_{k+1}.

$\therefore\ P'(0) = Dp_k$

$P'(1) = Dp_{k+1}$

Hence boundary condition for Hermite curve

$$P(0) = p_k$$
$$P(1) = p_{k+1} \quad \text{.................} \ \text{(ii)}$$
$$P'(0) = Dp_k$$
$$P'(1) = Dp_{k+1}$$

Matrix equivalent of eq. (i) is

$$P(u) = [u^3 \quad u^2 \quad u \quad 1]\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Similarly derivative of point function can be represented as,

$$P'(u) = [3u^2 \quad 2u \quad 1 \quad 0]\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

In matrix form, the Hermite boundary condition from eq. (ii) can be represented as

$$\begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Solving eq. (iv) for polynomial coefficient, we have

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} . \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix}$$

$$= M_H . \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix}$$

Where, M_H is the Hermite matrix.

Hence eq. (iii) can be represented as

$$P(u) = [u^3 \quad u^2 \quad u \quad 1].M_H.\begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} \text{ .... (v)}$$

Expanding (v)

$$P(u) = p_k(2u^3 - 3u^2 + 1) + p_{k+1}(-2u^3 + 3u^2) + Dp_k(u^3 - 2u^2 + u) + Dp_{k+1}(u^3 - u^2)$$

In terms of Hermite blending function, 'H', the Hermite curve can be represented as:

$$\boxed{P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u)}$$

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7. Bezier Curve

Bezier curve is developed by the French engineer Pierre Bezier for the design of Renault automobile bodies.

- It is an approximating spline widely used in various CAD system.
- Bezier curve is generated under the control of points known as control points.

General Bezier curve for (n+1) control point, denoted as $p_k = (x_k, y_k, z_k)$ with 'k' varying from 0 to n is given by

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u), \qquad 0 \le u \le 1 \quad \dots\dots\dots\dots (i)$$

Where, $P(u)$ is a point on Bezier Curve.

p_k is a control point.

$BEZ_{k,n}(u)$ is a Bezier blending function also known as *Bernstein Polynomial*.

Bezier blending function is defined as

$$BEZ_{k,n}(u) = C(n,k)u^k(1-u)^{n-k}$$

Where, $C(n,k) = \frac{n!}{K!(n-k)!}$

Individual x, y, z coordinates an a Bezier curve is given by,

$$x(u) = \sum_{k=0}^{n} x_k BEZ_{k,n}(u)$$

$$y(u) = \sum_{k=0}^{n} y_k BEZ_{k,n}(u)$$

$$z(u) = \sum_{k=0}^{n} z_k BEZ_{k,n}(u)$$

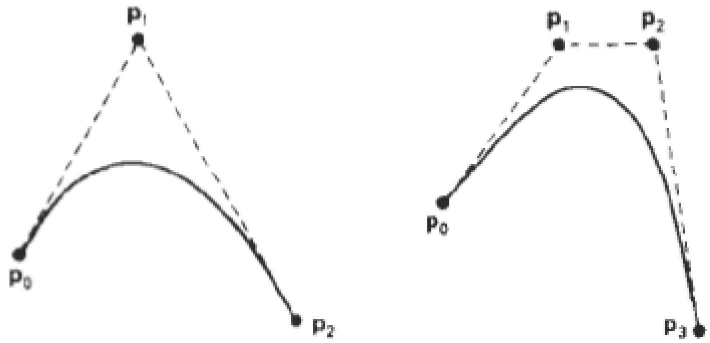


Fig: Bezier curve generated by three and four control point.

Properties of Bezier curve:

a) The basis functions are real.
b) The Bezier curve always passes through first and last control point i.e. $p(0) = p_0$ & $p(1) = p_n$.
c) The degree of polynomial representing Bezier curve is one less than the number of control points.
d) The Bezier curve always follows convex hull formed by control points.
e) The Bezier curve always lies inside the polygon formed by control points.
f) Bezier blending functions are positive and sum is equal to 1.
$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$
g) The direction of the tangent vector at the end points is same like vector determined by first and last segment.

Cubic Bezier Curve

- It is a Bezier curve generated by four control points.
- General equation for cubic Bezier curve is

$$P(u) = \sum_{k=0}^{3} p_k BEZ_{k,n}(u), \qquad 0 \le u \le 1 \quad \ldots\ldots\ldots\ldots (i)$$

$$P(u) = p_0 BEZ_{0,3}(u) + p_1 BEZ_{1,3}(u) + p_2 BEZ_{2,3}(u) + p_3 BEZ_{3,3}(u)$$

Where,

$$BEZ_{0,3}(u) = C(3,0)u^0(1-u)^{3-0} = \frac{3!}{0!\,(3-0)!} \times (1-u)^3 = (1-u)^3$$

Similarly,

$$BEZ_{1,3}(u) = 3u(1-u)^2$$
$$BEZ_{2,3}(u) = 3u^2(1-u)$$
$$BEZ_{3,3}(u) = u^3$$

$$\boxed{\therefore \; P(u) = p_0(1-u)^3 + p_1 3u(1-u)^2 + p_2 3u^2(1-u) + p_3 u^3}$$

In matrix form,

$$P(u) = [u^3 \quad u^2 \quad u \quad 1].M_{BEZ}.\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Where,

$$M_{BEZ} = Bezier\ matrix = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bezier surfaces

- Generalizations of Bezier curves to higher dimensions are called Bezier surfaces.
- The parametric vector function for the Bezier surface is formed as the Cartesian product of Bezier blending function:

$$P(u,v) = \sum_{j=0}^{m}\sum_{k=0}^{n} p_{j,k} BEZ_{j,m}(v) BEZ_{k,n}(u)$$

  With $p_{j,k}$ specifying the location of the (m+1) by (n+1) control points.
- Bezier surfaces have the same properties as Bezier curves, and they provide a convenient method for interactive design applications.

Q. Construct Bezier curve for control points (4, 2), (8, 8) and (16, 4).

Solution:

Given control points

$p_0 = (x_0, y_0) = (4,2)$

$p_1 = (x_1, y_1) = (8,8)$

$p_2 = (x_2, y_2) = (16,4)$

Here, degree (or order) n= 2

We have basis function as

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u), \qquad 0 \le u \le 1$$

$$= \sum_{k=0}^{2} p_k BEZ_{k,2}(u), \qquad 0 \le u \le 1$$

$$\therefore\ P(u) = p_0 BEZ_{0,2}(u) + p_1 BEZ_{1,2}(u) + p_2 BEZ_{2,2}(u)$$

Parametric equations are;

$$x(u) = x_0 BEZ_{0,2}(u) + x_1 BEZ_{1,2}(u) + x_2 BEZ_{2,2}(u) \ldots\ldots\ldots\ldots \text{(i)}$$
$$y(u) = y_0 BEZ_{0,2}(u) + y_1 BEZ_{1,2}(u) + y_2 BEZ_{2,2}(u) \ldots\ldots\ldots\ldots \text{(ii)}$$

Now,

$$BEZ_{0,2}(u) = C(2,0)u^0(1-u)^{2-0} = \frac{2!}{0!(2-0)!} \times (1-u)^2 = (1-u)^2$$
$$BEZ_{1,2}(u) = C(2,1)u^1(1-u)^{2-1} = \frac{2!}{1!(2-1)!} \times u(1-u)^1 = 2u(1-u)$$
$$BEZ_{2,2}(u) = C(2,2)u^2(1-u)^{2-2} = \frac{2!}{2!(2-2)!} \times u^2(1-u)^0 = u^2$$

Putting these values in eq. (i) & (ii) we get;

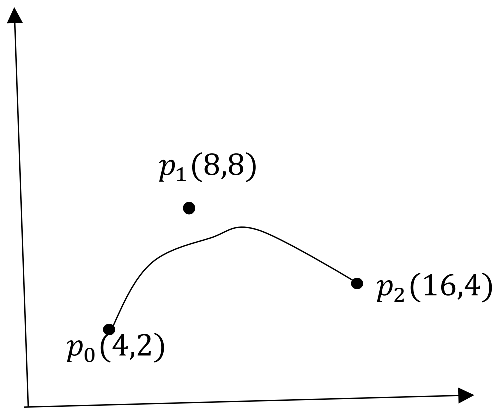
$x(u) = x_0(1-u)^2 + x_1 2u(1-u) + x_2 u^2 = 4u^2 + 8u + 4$

$y(u) = y_0(1-u)^2 + y_1 2u(1-u) + y_2 u^2 = -10u^2 + 12u + 2$

Now,

	$x(u)$	$y(u)$
u =0	4	2
u =0.2	5.76	4.0
u =0.4	7.84	5.20
u =0.6	10.24	5.6
u =0.8	12.96	5.2
u =1	16	4

Drawing these points we get:



Q. *Construct the Bezier curve of order 3 with 4 vertices of the control polygon $p_0(0,0)$, $p_1(1,2)$, $p_2(3,2)$& $p_3(2,0)$. Generate at least 5 points on the curve.*

Solution:

Given control points

$p_0 = (x_0, y_0) = (0,0)$

$p_1 = (x_1, y_1) = (1,2)$

$p_2 = (x_2, y_2) = (3,2)$

$p_3 = (x_3, y_3) = (2,0)$

Here, degree n= 3

We have basis function as

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u), \qquad 0 \le u \le 1$$

$$= \sum_{k=0}^{3} p_k BEZ_{k,3}(u), \qquad 0 \le u \le 1$$

$$\therefore\ P(u) = p_0 BEZ_{0,3}(u) + p_1 BEZ_{1,3}(u) + p_2 BEZ_{2,3}(u) + p_3 BEZ_{3,3}(u)$$

Parametric equations are;

$$x(u) = x_0 BEZ_{0,3}(u) + x_1 BEZ_{1,3}(u) + x_2 BEZ_{2,3}(u) + x_3 BEZ_{3,3}(u) \ \ldots\ldots\ldots\ldots \text{(i)}$$
$$y(u) = y_0 BEZ_{0,3}(u) + y_1 BEZ_{1,3}(u) + y_2 BEZ_{2,3}(u) + y_3 BEZ_{3,3}(u) \ \ldots\ldots\ldots\ldots \text{(ii)}$$

Now,

$$BEZ_{0,3}(u) = C(3,0)u^0(1-u)^{3-0} = \frac{3!}{0!(3-0)!} \times (1-u)^3 = (1-u)^3$$
$$BEZ_{1,3}(u) = C(3,1)u^1(1-u)^{3-1} = \frac{3!}{1!(3-1)!} \times u(1-u)^2 = 3u(1-u)^2$$
$$BEZ_{2,3}(u) = C(3,2)u^2(1-u)^{3-2} = \frac{3!}{2!(3-2)!} \times u^2(1-u)^1 = 3u^2(1-u)$$
$$BEZ_{3,3}(u) = C(3,3)u^3(1-u)^{3-3} = \frac{3!}{3!(3-3)!} \times u^3(1-u)^0 = u^3$$

Putting these values in eq. (i) & (ii) we get;

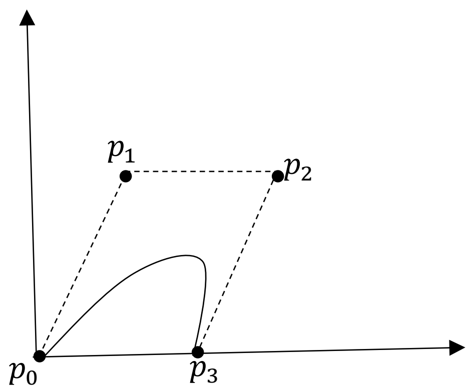
$$x(u) = x_0(1-u)^3 + x_1 3u(1-u)^2 + x_2 3u^2(1-u) + x_3 u^3 = -4u^3 + 3u^2 + 3u$$

$$y(u) = y_0(1-u)^3 + y_1 3u(1-u)^2 + y_2 3u^2(1-u) + y_3 u^3 = -6u^2 + 6u$$

Now,

	$x(u)$	$y(u)$
u =0	0	0
u =0.15	0.50	0.76
u =0.35	1.24	1.36
u =0.5	1.75	1.50
u =0.65	2.12	1.36
u =0.85	2.14	0.76
u =1	2	0

Plotting the graph;



8. B-spline Curve

B-spline curve is a set of piecewise polynomial segments that passes close to a set of control points.

It has two advantage over Bezier curve:

a) The degree of B-spline polynomial can be set independently of the number of control points.
b) It allows local control over the shape of a spline curve.

General equation of B-spline curve is given by

$$P(u) = \sum_{k=0}^{n} p_k B_{k,d}(u), \qquad 0 \le u \le n+d,\ \ 2 \le d \le n+1$$

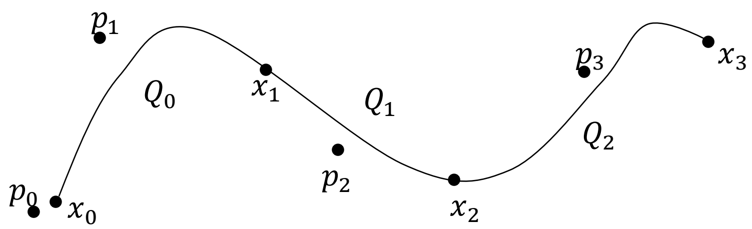
Where, p_k is a set of (n+1) control points.

$B_{k,d}(u)$ is the B-spline blending function.

Blending function for B-spline curves are defined by

$$B_{k,1}(u) = \begin{cases} 1, & if\ u_k \le u < u_{k+1} \\ 0, & Otherwise \end{cases}$$

$$B_{k,d}(u) = \frac{u-u_k}{u_{k+d-1}-u_k} B_{k,d-1}(u) + \frac{u_{k+d}-u}{u_{k+d}-u_{k+1}} B_{k+1,d-1}(u)$$



p_0, p_1, p_2, p_3 → Control point

x_0, x_1, x_2, x_3 → Knot values

Q_0, Q_1, Q_2 → Curve segment

- The knots produce a vector that defines the domain of the curve.

<u>Properties of B-Spline curve</u>

a) The polynomial curve has $d-1$ degree.
b) For (n+1) control points, the curve is described with (n+1) blending function.
c) Each blending function $B_{k,d}$ is defined over 'd' sub-interval of the total range of 'u', starting at knot value u_k.
d) The sum of B-spline basis functions for any parameter value is 1

$$\sum_{k=0}^{n} B_{k,d}(u) = 1$$

e) Each basis function is positive or zero for all parameter value.
f) The range of parameter 'u' is divided into (n+d) sub interval by (n+d+1) values specified in knot vector.
g) Each section of spline curve is influenced by 'd' control point.
h) Any one control point can affect the shape of at most 'd' curve sections.
i) The maximum order of curve is equal to the number of vertices of defining polygon.
j) The curve generally follows the shape of defining polygon.
k) The degree of B-spline polynomial is independent on the number of vertices of defining polygon.

On the basis of the knot points and interval length of segment there are two types of spline;

- **Periodic B-spline:** Knot points are equi-space to each other and splines are generated through the set of the equi-interval segments then such splines are called periodic B-splines.
- **Aperiodic B-spline:** If knot points are not equi-space to each other and splines are not generated through the set of the equi interval segments then such splines are called aperiodic B-splines.

Knot vector

There are three general classifications for knot vectors: uniform, open uniform and non-uniform.

- ***Uniform, periodic B-splines:***

When the spacing between knot values is constant, the resulting curve is a uniform B-spline. For e.g. {0, 1, 2, 3, 4, 5}

- Uniform B-splines have periodic blending function. That is, for given value of ‘n’ and ‘d’, all blending functions have the same shape
- Periodic splines are particularly useful for generating certain closed curves.

- ***Open uniform B-splines:***

For open B-splines, the knot spacing is uniform expect at the ends where knot values are repeated ‘d’ times. For e.g.

{0, 0, 1, 2, 3, 3} for d=2, and n=3

- ***Non-uniform B-splines:***

For non-uniform B-splines, we can choose multiple internal knot values and unequal spacing between the knot values. For e.g.

{0, 1, 2, 3, 3, 4}

{0, 0.2, 0.6, 0.9, 1.0}

- Non-uniform B-splines provide increased flexibility in controlling a curve shape.

References

- **Donald Hearne and M.Pauline Baker**, “Computer Graphics, C Versions.” Prentice Hall